III Semester M.Sc. Degree Examination, December 2014
(RNS) (Y2K11 Scheme)
MATHEMATICS
M303 : Differential Geometry

Time : 3 Hours Max. Marks : 80

Instructions : 1) *Answer **any five** questions choosing atleast **two** from **each** Part.*
2) ***All** questions carry **equal** marks.*

PART – A

1. a) Define directional derivative of a differentiable function on E^3.

If $V_p = (v_1, v_2, v_3)_p$ in any tangent vector to E^3 at $P \in E^3$, then prove that for any real valued differentiable function f on E^3, directional derivative of f by V_p in $V_p[f] = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_2}(p)$.

Use the above the formula to compute Vp [f]

for f = e^x cosy, Vp : P = (2, 0, –1), V = (2, –1, 3). **10**

b) Let h(s) = log s on J : s > 0. Reparametrize the curve $\alpha(t) = (e^t, e^{-t}, \sqrt{2}\ t$ using h. Verify the formula $B'(s) = \alpha'(h(s)) \frac{dh}{ds}(s)$, were B in a reparametrisation of α by h. **6**

2. a) Let f (x, y, z) = $(x^2 – 1)$ dy + $(y^2 + 2)$ z. Find the 1 – form df and evaluate it an V_p : V = (1, 2, –3), P = (0, –2, 1). **5**

b) Let $\phi = yzdx + dz$, $\psi = \sin z\ dz + \cos z\ dy$ be two 1 – forms on E^3. Then compute $\phi \wedge \psi$ and $d(\phi \wedge \psi)$. Verify the formula $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$. **6**

c) Let F : $E^3 \to E^3$ be a mapping defined by F (x, y, z) = (x cosy, x sin y, z). Then compute $F_{*p} V_p$ for Vp : V = (2, –1, 3), P = (0, 0, 1). **5**

P.T.O.

3. a) Derive Frenet formulae for a unit speed curve. **5**

b) Compute the Frenet apparatus T, N, B, K, T for a unit speed curve

$\beta(s) = \left(\frac{4}{5}\cos s,\ 1 - \sin s, \frac{-3}{5}\cos s\right)$ and show that it is a circle. **5**

c) With usual notations derive the formula $\nabla v_p W = \sum V_p[w_i]\ U_i\ (p)$, where $W = (w_1, w_2, w_3)$ is a vector field on E^3. Use it to compute $\nabla v_p W$ for $W = x^2 V_1 + y V_2$ and $V_p : V = (1, -1, 2)$ and $P = (1, 3, -1)$. **6**

4. a) Let E_1, E_2, E_3 be a frame field on E^3 with the attitude matrix $A = (a_{ij})$. Then show that the matrix of connection forms $W = (W_{ij})$ of E_1, E_2, E_3 is given by $W = (dA)\ A^t$, where dA is the differential and A^t is the transpose of A. Use the formula to compute connection forms of a cylindrical frame field. **9**

b) Prove the following :

i) an orthogonal transformation on E^3 is an isometry.

ii) an isometry $F : E^3 \to E^3$ with $F(0) = 0$ is an orthogonal transformation. **7**

PART – B

5. a) Define :

i) Proper patch

ii) Simple surface.

If $X : E^2 \to E^3$ is defined by X (u, v) = (u + v, u – v, uv), then prove that X is a proper patch in E^3 and its image is the surface $z = \frac{x^2 - y^2}{4}$. **6**

b) Let g be a real valued differentiable function on E^3 and 'c' be a real number. Show that the subset $M = \{(x, y, z) \in E^3 / g(x, y, z) = c\}$ is a surface in E^3 if $dg \neq 0$ at any point of M. Hence deduce that a sphere is a surface in E^3. **(6+4)**

6. a) Obtain parametrization of a cylinder. **4**

b) Let P be any point in a surface M and X be a patch in M with $P = X(u_0, v_0)$. Show that a tangent vector V_p to E^3 at P is tangent to M at P if and only if V_p is a linear combination of $X_u (u_0, v_0)$ and $X_u (u_0, v_0)$. **6**

c) With usual notations, prove

i) $F^*(\xi \wedge \eta) = F^*(\xi) \wedge F^*\eta$

ii) $F^*(d\xi) = d(F^*\xi)$. **6**

7. a) Define shape operator of a surface at a point. Find the shape operator of the saddle surface at (0, 0, 0). **6**

b) Show that every point on a sphere is a umbilic point. **6**

c) With usual notations prove $K = k_1k_2$, $H = \frac{1}{2}(k_1 + k_2)$. **4**

8. a) Prove that the Gaussian and mean curvatures of a surface are given by **4**

$$K = \frac{\begin{vmatrix} Sv.v & Sv.w \\ Sw.v & Sw.w \end{vmatrix}}{\begin{vmatrix} v.v & v.w \\ w.v & w.w \end{vmatrix}}, \quad H = \frac{\begin{vmatrix} Sv.v & Sv.w \\ w.v & w.w \end{vmatrix} + \begin{vmatrix} v.v & v.w \\ Sw.v & Sw.w \end{vmatrix}}{2\begin{vmatrix} v.v & v.w \\ w.v & w.w \end{vmatrix}}$$

b) If X is a patch in a surface M in E^3, then prove that the fundamental magnitudes l, m, n are given by $l = U.X_{uu}$, $M = U.X_{uv}$, $n = U.X_{vv}$, where U is a unit normal vector field on M. Hence compute the Gaussian and mean curvatures of $X(u, v) = (u \cos v, u \sin v, bv)$, $b \neq 0$. **7**

c) Determine the geodesics in **5**

i) plane

ii) sphere.